7 Additional Problems and Material

This supplemental chapter contains questions, definitions, and theorems related to previous ones, and it also contains new material.

7.1 Additional Material for Chapter 1

Example 7.1.

Let X denote the set of all lines in $\mathbb{R}^2$ that pass through the origin. Topologize X in a meaningful way – i.e. using neither the discrete or trivial topologies.

Example 7.2.

Let X denote the set of all lines in $\mathbb{R}^3$ that pass through the origin. Topologize X in a meaningful way – i.e. using neither the discrete or trivial topologies.

Example 7.3.

Let X denote the set of all $(n-1)$-dimensional vector subspaces of $\mathbb{R}^n$. Topologize X in a meaningful way – i.e. using neither the discrete or trivial topologies.

Definition A *metric* or *distance function* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:

- $\forall\, x, y \in X,\ d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$.
- $\forall\, x, y \in X,\ d(x, y) = d(y, x)$
- $\forall\, x, y, z \in X,\ d(x, z) + d(y, z) \leq d(x, y)$

A set X equipped with a metric d is referred to as a *metric space*.

Theorem 7.4. *Let (X, d) be a metric space. Then the sets*

$$N(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$$

can be used as a basis for a topology on X (here $x \in X$ and $\epsilon \in \mathbb{R}$ is positive).

Proof.

Prove the theorem $\square$

Example 7.5.

Let X be any set. Find a metric d so that the basis from Theorem 7.4 induces the discrete topology on X, or explain why no such metric exists. Do the same for the trivial topology on X.

Example 7.6.

Recall that the circle (or "1-sphere") is the space

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\}$$

equipped with the subspace topology of $\mathbb{E}^2$. Similarly, the "n-sphere" is the space

$$S^n = \left\{\vec{x} \in \mathbb{R}^{n+1} : ||\vec{x}||^2 = 1\right\}$$

equipped with the subspace topology in $\mathbb{E}^{n+1}$. Draw pictures of S^1 and S^2 and indicate at least one, proper open subset in each space. Describe how one might visualize S^3.

Theorem 7.7. *(Pasquale's Theorem) A space X has the discrete topology $\iff$ $\forall x \in X$ the singleton $\{x\}$ is open.*

Proof.

Prove or disprove. $\square$

Theorem 7.8. *Suppose Y is a subspace of X. Then a set $C \subseteq Y$ is closed $\iff$ $C = K \cap Y$ where K is clsoed in X.*

Proof.

Prove or disprove. $\square$

Example 7.9. Let X be a set and define

$$\mathcal{T} = \{S \subseteq X : S^c \text{ is countable }\} \cup \{\emptyset\}.$$

Prove that $\{X, \mathcal{T}\}$ is a topological space. This topology is called the *cocountable* or *countable complement* topology, and when $X = \mathbb{R}$ is equipped with it, we will use the symbol $\mathbf{C^1}$.

7.2 Additional Material for Chapter 2

Theorem 7.10. *Suppose f, g are continuous functions from $\mathbb{E}^n$ to $\mathbb{E}^1$. Prove that $p(\vec{x}) = f(\vec{x}) \cdot g(\vec{x})$ is also continuous. Prove that $f(\vec{x}) \pm g(\vec{x})$ is also continuous.*

Proof.

Prove the theorem. Is this result true if one uses *other* topologies on $\mathbb{R}$? $\square$

Example 7.11.

Consider the function $D : \mathbb{R}^4 \to \mathbb{R}$ given by $D(a, b, c, d) = ad - bc$. With respect to the Euclidean topologies for both $\mathbb{R}^4$ and $\mathbb{R}$, prove that D is continuous.

Example 7.12.

Let $\mathrm{GL}_2(\mathbb{R})$ denote the set of all invertible 2×2 matrices. Topologize $\mathrm{GL}_2(\mathbb{R})$ in a meaningful way – i.e. using neither the discrete or trivial topologies.

Definition An *embedding* is a map $f : X \to Y$ that is a homeomorphism onto its image; i.e. a map for which $X \approx f(X)$. Here we regard the subset $f(X) \subseteq Y$ as being equipped with the subspace topology.

Example 7.13.

Let S^n denote the n-sphere (as defined in the previous subsection), and let $k < m$ be natural numbers. Find an embedding $f : S^k \to S^m$, and prove your function is an embedding.

Example 7.14.

Find an embedding $f : \mathbb{R}^n \to S^n$.

Example 7.15.

Let $SO(2)$ denote the set of matrices $A \in \mathrm{GL}_2(\mathbb{R})$ satisfying $\det A = 1$ and $A\,A^T = I$. Explain how $SO(2) \approx S^1$, where $SO(2)$ is equipped with the subspace topology from the topology defined in 7.9. Can you prove that these two spaces are homeomorphic? Is it true that $SO(3) \approx S^2$?

7.3 Additional Material for Chapter 3

Theorem 7.16. $\mathbf{C^1}$ *is connected.*

Proof.

Prove or disprove. □

Theorem 7.17. $\mathbf{C^1}$ *is Hausdorff.*

Proof.

Prove or disprove. □

Definition A sequence of points $\{x_n \in X : n \in \mathbb{N}\}$ in a topological space X is said *to converge* to the point $x \in X$ if for every open set $U \subseteq X$ which contains x, there exists $N \in \mathbb{N}$ such that $\{x_n : n \geq N\} \subseteq U$.

Example 7.18.

Let X be a trivial topological space. Explain why / prove every sequence converges to *every* point $x \in X$.

Example 7.19.

Let $X = \mathbf{F^1}$ and consider the sequence $\{n : n \in \mathbb{N}\}$. To which points, if any, does this sequence converge? Determine all (other?) divergent sequences.

Theorem 7.20. *Let X be a Hausdorff space. If the sequence $\{x_n\}$ converges to x and y, then ...*

Proof.

Complete and prove the theorem. □

Theorem 7.21. *State the converse to the above theorem.*

Proof.

Prove or disprove. □

Definition A space X is said to be *locally compact* if for every point $x \in X$ there exists a compact, open set U that contains x.

Theorem 7.22. *Every compact space is a locally compact space.*

Proof.

Prove or disprove, but oh my god, if you disprove it then go away. □

Example 7.23. Explain why $\mathbb{Q}$, endowed with the subspace topology from $\mathbb{E}$, is *not* locally compact.

Theorem 7.24. *All open and closed subsets of a locally compact, Hausdorff space X are locally compact.*

Proof.

Prove or disprove. ☐

Definition A *compactification* of a topological space X is a topological space Y that contains X as a subspace

Definition Suppose X is a topological space with topology $\mathcal{T}$. Let ∞ denote some abstract point that is not in X, and let $\hat{X} = X \cup \{\infty\}$. Define a topology $\hat{\mathcal{T}}$ on $\hat{X}$ by declaring $U \subseteq \hat{X}$ to be open if either

1. $\infty \notin U$ and $U \in \mathcal{T}_X$ or
2. $\infty \in U$ and $X - U$ is a closed, compact subset of X.

The new space $\{\hat{X}, \hat{\mathcal{T}}\}$ is called *the one point compactification of* X.

Theorem 7.25. $\{\hat{X}, \hat{\mathcal{T}}\}$ *is a topological space.*

Proof.

Prove. ☐

Theorem 7.26. *Endow X with the subspace topology it inherits from $\mathcal{X}$. This topology coincides with the original topology $\mathcal{T}$ on X.*

Proof.

Do the opposite of disprove. ☐

Theorem 7.27. *Given any space X, the one point compactification $\hat{X}$ is compact.*

Proof.

Prove. ☐